

Extremes in Random Graphs Models of Complex Networks

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Abstract. Regarding the analysis of Web communication, social and complex networks the fast finding of most influential nodes in a network graph constitutes an important research problem. We use two indices of the influence of those nodes, namely, PageRank and a Max-linear model. We consider the PageRank as an autoregressive process with a random number of random coefficients that depend on ranks of incoming nodes and their out-degrees and assume that the coefficients are independent and distributed with regularly varying tail and with the same tail index. Then it is proved that the tail index and the extremal index are the same for both PageRank and the Max-linear model and the values of these indices are found. The achievements are based on the study of random sequences of a random length and the comparison of the distribution of their maxima and linear combinations.

Keywords: Extremal Index, PageRank, Max-Linear Model, Branching Process, Autoregressive Process, Complex Networks.

1 Introduction

Regarding the analysis of Web communication, social and complex networks the fast finding of most influential nodes in a network graph constitutes an important research problem. PageRank remains the most popular characteristic of such influence. We aim to find an extremal index of PageRank whose reciprocal value determines the first hitting time, i.e. a minimal time to reach the first influential node by means of a PageRank random walk. The extremal index $\theta \in [0, 1]$ has many other interpretations and plays a significant role in the theory of extreme values. Particularly, the limit distribution of maxima of stationary random variables (r.v.s) depends on θ . For independent r.v.s $\theta = 1$ holds.

θ has a connection to the tail index that shows the heaviness of the tail of a stationary distribution of an underlying process.

Google's PageRank defines the rank $R(X_i)$ of the Web page X_i as

$$R(X_i) = c \sum_{X_j \in N(X_i)} \frac{R(X_j)}{D_j} + (1 - c)q_i, \quad i = 1, \dots, n, \quad (1)$$

where $N(X_i)$ is the set of pages that link to X_i (in-degree), D_j is the number of outgoing links of page X_j (out-degree), $c \in (0, 1)$ is a damping factor, $q =$

(q_1, q_2, \dots, q_n) is a personalization probability vector or user preference such that $q_i \geq 0$ and $\sum_{i=1}^n q_i = 1$, and n is the total number of pages, [11]. We omit in (1) the term with dangling nodes for simplicity.

PageRank of a randomly selected page (a node in the graph) with random in- and out-degrees may be considered as a branching process (Cf. [4], [5], [14])

$$R_i = \sum_{j=1}^{N_i} A_j R_i^{(j)} + Q_i, \quad i = 1, \dots, n, \quad (2)$$

denoting $R_i = R(X_i)$, $A_j =^d c/D_j$, $Q_i = (1 - c)q_i$, [14]. $R_i^{(j)}$ are ranks of descendants of node i , i.e. nodes with incoming links to node i . The r.v. N_i determines an in-degree, i.e. a number of directed edges to the i th node, and a number of nodes in the first generation of descendants belonging to the i th node as a parent, $\{Q_i\}$ is a sequence of i.i.d. r.v.s.

Starting from the initial page (node) X_0 , a PageRank random walk determines a regenerative process or Harris recurrent process $\{X_t\}$, letting it visits pages-followers of the underlying node with probability c and it restarts with probability $1 - c$ by jumping to a random independent node.

A Max-linear model can be considered as an alternative characteristic of the node influence. This model is obtained by a substitution of sums in Google's definition of PageRank by maxima, i.e.

$$R_i = \bigvee_{j=1}^{N_i} A_j R_i^{(j)} \vee Q_i, \quad i = 1, \dots, n, \quad (3)$$

is proposed in [6].

Formally, (2) can be considered as an autoregressive process with the random number N_i of random coefficients and the independent random term Q_i . The extremal index of $AR(1)$ processes with regularly varying stationary distribution and its relation to the tail index were considered in [9]. The extremal index of $AR(q)$, $q \geq 1$ processes with q random coefficients was obtained in [10] in a form which is not convenient for calculations. In [7] the results by [9] were extended to multivariate regularly varying distributed random sequences and the extremal and tail indices of sum and maxima of such sequences with $l \geq 1$ r.v.s were derived.

Our achievements extend and adapt the results by [7] to PageRank and Max-linear processes. The problem concerns the finding of the extremal index of a random graph that models a real network where incoming nodes of the root node may be linked and, hence, be dependent. Such a random graph is called a Thorny Branching Tree (TBT) since any node may have outbound stubs (teleportations) to arbitrary nodes of the network, [4]. In this respect, such a graph cannot be considered as a pure Galton-Watson branching process where descendants of any node are mutually independent and teleportations are impossible.

The paper is organized as follows. In Section 2 we recall necessary results regarding the relation between the tail and extremal indices obtained in [7] for

multivariate random sequences which are regularly varying distributed (Theorems 1 and 2). Linear combinations and maxima of the random sequences of a fixed length are considered and it is derived that they have the same tail and extremal indices. In Section 3 we extend Theorem 2 to the case of unequal tail indices assuming r.v.s of a random sequence (Theorem 3). In Section 4 we consider sequences of random lengths and obtain the tail and extremal indices of their linear combinations and maxima (Theorem 4). We further discuss how these results can be applied to PageRank and the Max-linear processes in Section 5.

2 Related Work

Let $\{R_j\}$ be a stationary sequence with distribution function $F(x)$ and maxima $M_n = \max_{1 \leq j \leq n} R_j$. We shall interpret $\{R_j\}$ as PageRanks of Web pages.

Definition 1. A stationary sequence $\{R_n\}_{n \geq 1}$ is said to have extremal index $\theta \in [0, 1]$ if for each $0 < \tau < \infty$ there is a sequence of real numbers $u_n = u_n(\tau)$ such that

$$\lim_{n \rightarrow \infty} n(1 - F(u_n)) = \tau \quad \text{and} \quad (4)$$

$$\lim_{n \rightarrow \infty} P\{M_n \leq u_n\} = e^{-\tau\theta} \quad (5)$$

hold ([12], p.53).

In [7] the following theorems are proved which we will use to find the extremal and tail indices of PageRank and a Max-linear model. Let $Y_n^{(1)}, Y_n^{(2)}, \dots, Y_n^{(l)}$, $n \geq 1$, $l \geq 1$ be sequences of r.v.s having stationary distributions with tail indices k_1, \dots, k_l and extremal indices $\theta_1, \dots, \theta_l$, respectively, i.e.

$$P\{Y_n^{(i)} > x\} \sim c^{(i)} x^{-k_i} \quad \text{as} \quad x \rightarrow \infty,$$

where $c^{(i)}$ are some real positive constants.

Let us consider the weighted sum

$$Y_n(z) = z_1 Y_n^{(1)} + z_2 Y_n^{(2)} + \dots + z_l Y_n^{(l)}, \quad z_1, \dots, z_l > 0 \quad (6)$$

and denote its tail index by $k(z)$ and extremal index by $\theta(z)$. Supposing that there is a minimal tail index among k_1, \dots, k_l , the following theorem states the corresponding $k(z)$ and $\theta(z)$.

Theorem 1. ([7]) *Let $k_1 < k_i$, $i = 2, \dots, l$ hold. Then $Y_n(z)$ has the tail index $k(z) = k_1$ and the extremal index $\theta(z) = \theta_1$.*

In the next theorem it is assumed that sequences $Y_n^{(1)}, Y_n^{(2)}, \dots, Y_n^{(l)}$ are mutually independent with equal tail indices $k_1 = \dots = k_l = k$. We denote

$$Y_n^*(z) = \max \left(z_1 Y_n^{(1)}, z_2 Y_n^{(2)}, \dots, z_l Y_n^{(l)} \right). \quad (7)$$

Theorem 2. ([7]) *The sequences $Y_n^*(z)$ and $Y_n(z)$ have the same tail index k and the same extremal index equal to*

$$\theta(z) = \frac{c^{(1)} z_1^k}{c^{(1)} z_1^k + \dots + c^{(l)} z_l^k} \theta_1 + \dots + \frac{c^{(l)} z_l^k}{c^{(1)} z_1^k + \dots + c^{(l)} z_l^k} \theta_l.$$

3 Generalization of Theorem 2

Theorem 3 is a generalization of Theorem 2 to the case of unequal tail indices.

Theorem 3. *Let $\{Y_n^{(j)}\}$, $n \geq 1$, $j = 1, \dots, l$ be mutually independent regularly varying r.v.s with tail indices k_1, \dots, k_l , respectively. Let $k_m < k_i$, $i = 1, \dots, l$, $i \neq m$ hold. Then r.v.s $Y_n^*(z)$ and $Y_n(z)$ have the same tail index $k(z) = k_m$ and the same extremal index $\theta(z) = \theta_m$.*

Proof. First we show that

$$P\{Y_n^*(z) > x\} \sim c(z)x^{-k_m}, \quad x \rightarrow \infty, \quad (8)$$

where $c(z) = \sum_{i=1}^l c^{(i)} z_i^{k_i} \mathbf{1}\{k_i = k_m\}$. Similar to [7] and as

$$P\{z_i Y_n^{(i)} > x\} \sim c^{(i)} z_i^{k_i} x^{-k_i} \quad (9)$$

holds, we have

$$\begin{aligned} P\{Y_n^*(z) > x\} &= P\{\max(z_1 Y_n^{(1)}, \dots, z_l Y_n^{(l)}) > x\} \\ &= 1 - P\{\max(z_1 Y_n^{(1)} \leq x) \cdot \dots \cdot P\{z_l Y_n^{(l)} \leq x\} \\ &= \sum_{i=1}^l P\{z_i Y_n^{(i)} > x\} \\ &\quad + \sum_{k=2}^l (-1)^{k-1} \sum_{i_1 < i_2 < \dots < i_k; i_1, i_2, \dots, i_k=1}^l P\{z_{i_1} Y_n^{(i_1)} > x\} \cdot \dots \cdot P\{z_{i_k} Y_n^{(i_k)} > x\} \\ &\sim \sum_{i=1}^l c^{(i)} z_i^{k_i} x^{-k_i} \\ &\quad + \sum_{k=2}^l (-1)^{k-1} \sum_{i_1 < i_2 < \dots < i_k; i_1, i_2, \dots, i_k=1}^l c^{(i_1)} z_{i_1}^{k_{i_1}} x^{-k_{i_1}} \cdot \dots \cdot c^{(i_k)} z_{i_k}^{k_{i_k}} x^{-k_{i_k}} \\ &\sim c(z)x^{-k_m} + o(x^{-k_m}), \quad x \rightarrow \infty. \end{aligned} \quad (10)$$

Thus, $P\{Y_n(z) > x\} \sim c(z)x^{-k_m}$ follows from Theorem 1.

Now we show that $Y_n^*(z)$ and $Y_n(z)$ have the same extremal index $\theta(z) = \theta_m$.

We use the same notations as in [7]

$$\begin{aligned} M_n^{(i)} &= \max\{Y_1^{(i)}, Y_2^{(i)}, \dots, Y_n^{(i)}\}, \quad i = 1, \dots, l; \\ M_n(z) &= \max\{Y_1(z), Y_2(z), \dots, Y_n(z)\}, \\ M_n^*(z) &= \max\{Y_1^*(z), Y_2^*(z), \dots, Y_n^*(z)\}, \quad n \geq 1. \end{aligned}$$

By (7) it holds

$$\begin{aligned} M_n^*(z) &= \max\{z_1 Y_1^{(1)}, \dots, z_1 Y_n^{(1)}, \dots, z_l Y_1^{(l)}, \dots, z_l Y_n^{(l)}\} \\ &= \max\{z_1 M_n^{(1)}, \dots, z_l M_n^{(l)}\}. \end{aligned}$$

Then we get

$$P\{M_n^*(z)n^{-1/k} \leq x\} = P\{z_1 M_n^{(1)} n^{-1/k} \leq x, \dots, z_l M_n^{(l)} n^{-1/k} \leq x\} \quad (11)$$

Since k_m is the minimal tail index we have

$$P\{z_i M_n^{(i)} n^{-1/k_m} \leq x\} = P\{z_i M_n^{(i)} n^{-1/k_i} \leq x n^{1/k_m - 1/k_i}\}.$$

It implies

$$z_i M_n^{(i)} n^{-1/k_m} \xrightarrow{P} 0, \quad i = 1, \dots, l, \quad i \neq m \quad \text{as } n \rightarrow \infty \quad (12)$$

since $\lim_{n \rightarrow \infty} P\{z_i M_n^{(i)} n^{-1/k_i} \leq x\} = \exp(-c^{(i)} \theta_i z_i^{k_i} x^{-k_i})$. By (11) it holds

$$P\{M_n^*(z) n^{-1/k_m} \leq x\} \rightarrow \exp(-c^{(m)} z_m^{k_m} \theta_m x^{-k_m}), \quad n \rightarrow \infty.$$

Now we have to show that $P\{M_n^*(z) n^{-1/k_m} \leq x\} \sim P\{M_n(z) n^{-1/k_m} \leq x\}$. Let us denote $u_n = x n^{1/k_m}$. Note that the event $\{M_n^*(z) \leq u_n\}$ follows from $\{M_n(z) \leq u_n\}$. Then, as in [7], we obtain

$$\begin{aligned} 0 &\leq P\{M_n^*(z) \leq u_n\} - P\{M_n(z) \leq u_n\} \\ &= P\{M_n^*(z) \leq u_n\} - P\{M_n^*(z) \leq u_n, M_n(z) \leq u_n\} \\ &= P\{M_n^*(z) \leq u_n, M_n(z) > u_n\} \leq \sum_{k=1}^n P\{M_n^*(z) \leq u_n, Y_k(z) > u_n\} \\ &\leq \sum_{k=1}^n P\{Y_k^*(z) \leq u_n, Y_k(z) > u_n\} = n P\{Y_n^*(z) \leq u_n, Y_n(z) > u_n\} \end{aligned} \quad (13)$$

due to the stationarity of the sequences $Y_n^*(z)$ and $Y_n(z)$. Lemma 1 in [7] states that

$$P\{Y_n^*(z) \leq u_n | Y_n(z) > u_n\} \rightarrow 0, \quad n \rightarrow \infty, \quad (14)$$

for i.i.d. regularly varying $\{Y_n^{(j)}\}$ with equal tail index. This can be extended to the case of unequal k_1, \dots, k_l . Since

$$n P\{Y_n(z) > u_n\} \rightarrow c(z) x^{-k_m}, \quad n \rightarrow \infty, \quad (15)$$

and (14) hold, it follows

$$\lim_{n \rightarrow \infty} (P\{M_n^*(z) \leq u_n\} - P\{M_n(z) \leq u_n\}) = 0.$$

4 Extremal Index of PageRank and the Max-Linear Processes

We denote in (2) R_i as $Y_i(z)$ and $A_j R_i^{(j)} = c R_i^{(j)} / D_j$, $j = 1, \dots, N_i$ as $z_j Y_i^{(j)}$. Then we can represent (2) in the form (6) as

$$Y_i(z) = \sum_{j=1}^{N_i} z_j Y_i^{(j)} + Q_i, \quad i = 1, \dots, n, \quad (16)$$

where N_i is a nonnegative integer-valued r.v.. In the context of PageRank $z_j = c$, $j = 1, 2, \dots, N_i$, $Q_i = z^* q_i$ with $z^* = 1 - c$ and N_i represents the node in-degree. It is realistic to assume that N_i is a power law distributed r.v. with parameter $\alpha > 0$, i.e.

$$P\{N_i = \ell\} \sim \ell^{-\alpha} \quad (17)$$

and N_i is bounded by a total number of nodes in the network.

The distribution of N_i is in the domain of attraction of the Fréchet distribution with shape parameter $\alpha > 0$ and $P\{N_i > x\} = x^{-\alpha} \ell(x)$, $\forall x > 0$, where $\ell(x)$ is a slowly varying function, since it satisfies a sufficient condition for this property, i.e. the von Mises type condition $\lim_{n \rightarrow \infty} nP\{N_i = n\}/P\{N_i > n\} = \alpha$, [1].

Theorem 4 is an extension of Theorems 2 and 3 to maxima and sums of multivariate random sequences of random lengths, that can be applied to PageRank and the Max-linear processes. Let us turn to (16) and denote

$$\begin{aligned} Y_{N_n}^*(z) &= \max(z_1 Y_n^{(1)}, \dots, z_{N_n} Y_n^{(N_n)}, Q_n), \\ Y_{N_n}(z) &= z_1 Y_n^{(1)} + \dots + z_{N_n} Y_n^{(N_n)} + Q_n. \end{aligned}$$

Theorem 4. Let $\{Y_n^{(j)}\}$, $n \geq 1$, $j = 1, \dots, N_n$ and $q_n = Q_n/z^*$ be mutually independent regularly varying i.i.d. r.v.s with tail indices $k > 0$ and $\beta > 0$, respectively, and N_n be regularly varying r.v. with tail index $\alpha > 0$. Let $Y_n^{(1)}, \dots, Y_n^{(N_n)}$ have extremal indices $\theta_1, \dots, \theta_{N_n}$, respectively. Then r.v.s $Y_{N_n}^*(z)$ and $Y_{N_n}(z)$ are regularly varying distributed with the same tail index $k(z) = \min(k, \alpha, \beta)$ and the same extremal index $\theta(z)$ such that

$$\begin{aligned} \theta(z) &= (z^*)^\beta, \quad \text{if } k \geq \beta, \\ \theta(z) &= \sum_{i=1}^{\infty} c^{(i)} \theta_i z_i^k / c(z), \quad \text{if } k < \beta, \end{aligned} \quad (18)$$

where $c(z) = \sum_{i=1}^{\infty} c^{(i)} z_i^k$ holds.

Proof. We shall show first that

$$P\{Y_{N_n}^*(z) > x\} \sim P\{Y_{N_n}(z) > x\} \sim x^{-\min(k, \alpha, \beta)}. \quad (19)$$

Since r.v.s $\{Y_n^{(j)}\}_{j \geq 1}$ are subexponential and i.i.d. it holds

$$\begin{aligned} P\{z_1 Y_n^{(1)} + \dots + z_{[x]} Y_n^{([x])} > x\} &\sim P\{\max(z_1 Y_n^{(1)}, \dots, z_{[x]} Y_n^{([x])}) > x\} \\ &\sim x P\{z_1 Y_n^{(1)} > x\}, \quad x \rightarrow \infty, \end{aligned} \quad (20)$$

[8]. Due to mutual independence of Q_n and $\{Y_n^{(j)}\}$ and similar to (10) we get

$$\begin{aligned} P\{Y_{N_n}^*(z) > x\} &= P\{Y_{N_n}^*(z) > x, N_n \leq x\} + P\{Y_{N_n}^*(z) > x, N_n > x\} \\ &\leq P\{Y_{[x]}^*(z) > x\} + P\{N_n > x\} \\ &= 1 - P\{\max(z_1 Y_n^{(1)}, \dots, z_{[x]} Y_n^{([x])}) \leq x\} P\{Q_n \leq x\} + P\{N_n > x\} \\ &\sim c_N x^{-\alpha} + c_q (z^*)^\beta x^{-\beta} + c(z) x^{-k} \sim x^{-\min\{k, \alpha, \beta\}}, \end{aligned} \quad (21)$$

as $x \rightarrow \infty$, where $c_N, c_q > 0$, $c(z) = \sum_{i=1}^{\infty} c^{(i)} z_i^k$. On the other hand,

$$\begin{aligned} & P\{Y_{N_n}^*(z) > x\} \geq 0 + P\{Y_{N_n}^*(z) > x, N_n > x\} \\ & \geq P\{Y_{\lceil x \rceil}^*(z) > x\} + P\{N_n > x\} + P\{Y_{\lceil x \rceil}^*(z) \leq x, N_n \leq x\} - 1 \\ & \sim x^{-\min\{k, \alpha, \beta\}} \end{aligned} \quad (22)$$

holds, since $P\{Y_{\lceil x \rceil}^*(z) \leq x, N_n \leq x\} \rightarrow 1$ as $x \rightarrow \infty$. Due to (21) and (22) we obtain

$$P\{Y_{N_n}^*(z) > x\} \sim x^{-\min\{k, \alpha, \beta\}}.$$

The same is valid for $Y_{N_n}(z)$ by substitution of the maximum by the sum due to (20). Hence, (19) follows.

Let us prove that $Y_{N_n}^*(z)$ and $Y_{N_n}(z)$ have the same extremal index $\theta(z)$. Let us denote

$$\begin{aligned} M_{N_n}^*(z) &= \max\{Y_{N_1}^*(z), Y_{N_2}^*(z), \dots, Y_{N_n}^*(z)\} \\ &= \max\{z_1 Y_1^{(1)}, \dots, z_{N_1} Y_1^{(N_1)}, Q_1, \dots, z_1 Y_n^{(1)}, \dots, z_{N_n} Y_n^{(N_n)}, Q_n\} \end{aligned} \quad (23)$$

and

$$\begin{aligned} M_{N_n}(z) &= \max\{Y_{N_1}(z), Y_{N_2}(z), \dots, Y_{N_n}(z)\} \\ &= \max\{z_1 Y_1^{(1)} + \dots + z_{N_1} Y_1^{(N_1)} + Q_1, \dots, z_1 Y_n^{(1)} + \dots + z_{N_n} Y_n^{(N_n)} + Q_n\}. \end{aligned}$$

Without loss of generality we may assume that $N_n = \max\{N_1, \dots, N_n\}$. Then we can complete vectors $(z_1 Y_i^{(1)}, \dots, z_{N_i} Y_i^{(N_i)})$, $i = 1, 2, \dots, n$ by zeros up to the dimension N_n and separate the vector (Q_1, \dots, Q_n) . We rewrite (23) as

$$\begin{aligned} M_{N_n}^*(z) &= \max\{z_1 Y_1^{(1)}, \dots, z_1 Y_n^{(1)}, \dots, z_{N_n} \cdot 0, \dots, z_{N_n} \cdot 0, \dots, z_{N_n} Y_n^{(N_n)}, \\ & Q_1, \dots, Q_n\} \\ &= \max(z_1 M_n^{(1)}, z_2 M_n^{(2)}, \dots, z_{N_n} M_n^{(N_n)}, M_n^{(Q)}). \end{aligned}$$

Here, $M_n^{(Q)} = \max\{Q_1, \dots, Q_n\}$ relates to the second term in the rhs of (16) corresponding to the user preference term Q_i in (2). Following the same arguments as after (11) in Section 3 the statement follows. Really, denoting $k^* = \min\{k, \beta\}$ and $u_n = x n^{1/k^*}$, $x > 0$, we get

$$\begin{aligned} & P\{M_{N_n}^*(z) > u_n\} \\ &= P\{M_{N_n}^*(z) > u_n, N_n > u_n\} + P\{M_{N_n}^*(z) > u_n, N_n \leq u_n\} \\ &\leq P\{M_{\lceil u_n \rceil}^*(z) > u_n\} + P\{N_n > u_n\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} & P\{M_{N_n}^*(z) > u_n\} \geq P\{M_{\lceil u_n \rceil}^*(z) > u_n, N_n > u_n\} \\ &= P\{N_n > u_n\} + P\{M_{\lceil u_n \rceil}^*(z) > u_n\} + P\{M_{\lceil u_n \rceil}^*(z) \leq u_n, N_n \leq u_n\} - 1. \end{aligned}$$

Note that $P\{M_{\lfloor u_n \rfloor}^*(z) \leq u_n, N_n \leq u_n\} - 1$ tends to zero as $n \rightarrow \infty$. Hence, it holds

$$P\{M_{N_n}^*(z) > u_n\} \sim P\{M_{\lfloor u_n \rfloor}^*(z) > u_n\} + P\{N_n > u_n\}, \quad n \rightarrow \infty. \quad (24)$$

If $k < \beta$ holds, then $M_n^{(Q)} \cdot n^{-1/k^*} \xrightarrow{P} 0$ as $n \rightarrow \infty$ since $P\{z_i M_n^{(i)} n^{-1/k} \leq x\} \rightarrow \exp(-c^{(i)} \theta_i z_i^k x^{-k})$, $i = 1, 2, \dots$. Since $P\{N_n > u_n\} \sim u_n^{-\alpha} \rightarrow 0$ as $n \rightarrow \infty$ holds, then by (24) it follows

$$\lim_{n \rightarrow \infty} P\{M_{N_n}^*(z) n^{-1/k^*} \leq x\} = \exp\{-c(z) \theta^*(z) x^{-k}\}, \quad (25)$$

where $\theta^*(z) = \sum_{i=1}^{\infty} c^{(i)} \theta_i z_i^k / c(z)$ and $c(z) = \sum_{i=1}^{\infty} c^{(i)} z_i^k$.

If $k \geq \beta$ holds, then $z_i M_n^{(i)} \cdot n^{-1/k^*} \xrightarrow{P} 0$, $i = 1, 2, \dots$ follows since $P\{M_n^{(Q)} n^{-1/\beta} \leq x\} \rightarrow \exp(-c_q(z^*)^\beta x^{-\beta})$ as $n \rightarrow \infty$. Thus, we obtain

$$\lim_{n \rightarrow \infty} P\{M_{N_n}^*(z) n^{-1/k^*} \leq x\} = \exp(-c_q(z^*)^\beta x^{-\beta}). \quad (26)$$

Since $\{q_i\}$ are i.i.d., its extremal index is equal to one. Then by (25) and (26) the extremal index of $Y_{N_n}^*(z)$ satisfies (18) irrespectively of α .

It remains to show that $Y_{N_n}^*(z)$ and $Y_{N_n}(z)$ have the same extremal index. Similarly to [7], we have to derive that

$$\lim_{n \rightarrow \infty} P\{M_{N_n}(z) n^{-1/k^*} \leq x\} = \lim_{n \rightarrow \infty} P\{M_{N_n}^*(z) n^{-1/k^*} \leq x\}. \quad (27)$$

Since from the event $\{M_{N_n}(z) \leq u_n\}$ it follows $\{M_{N_n}^*(z) \leq u_n\}$, and $P\{M_{N_n}(z) \leq u_n\} \leq P\{M_{N_n}^*(z) \leq u_n\}$ holds, we obtain similarly to (13)

$$\begin{aligned} 0 &\leq P\{M_{N_n}^*(z) \leq u_n\} - P\{M_{N_n}(z) \leq u_n\} \\ &= P\{M_{N_n}^*(z) \leq u_n\} - P\{M_{N_n}^*(z) \leq u_n, M_{N_n}(z) \leq u_n\} \\ &= P\{M_{N_n}^*(z) \leq u_n, M_{N_n}(z) > u_n\} \\ &= P\{M_{N_n}^*(z) \leq u_n, M_{N_n}(z) > u_n, N_n > u_n\} \\ &\quad + P\{M_{N_n}^*(z) \leq u_n, M_{N_n}(z) > u_n, N_n \leq u_n\} \\ &\leq P\{N_n > u_n\} + P\{M_{N_n}^*(z) \leq u_n, M_{\lfloor u_n \rfloor}(z) > u_n, N_n \leq u_n\} \\ &\leq P\{N_n > u_n\} + \sum_{k=1}^{\lfloor u_n \rfloor} P\{Y_k^*(z) \leq u_n, Y_k(z) > u_n\} \\ &= P\{N_n > u_n\} + \lfloor u_n \rfloor P\{Y_k^*(z) \leq u_n, Y_k(z) > u_n\} \end{aligned} \quad (28)$$

due to the stationarity of $\{Y_k^*(z)\}$ and $\{Y_k(z)\}$.

Completing vectors $(z_1 Y_k^{(1)}, \dots, z_{N_k} Y_k^{(N_k)})$ by zeroes up to the maximal dimension $\lfloor u_n \rfloor$, we get

$$\begin{aligned} P\{Y_k^*(z) \leq u_n, Y_k(z) > u_n\} &= P\{\max(z_1 Y_k^{(1)}, \dots, z_{\lfloor u_n \rfloor} Y_k^{(\lfloor u_n \rfloor)}, Q_k) \leq u_n, \\ &\quad z_1 Y_k^{(1)} + \dots + z_{\lfloor u_n \rfloor} Y_k^{(\lfloor u_n \rfloor)} + Q_k > u_n\} \end{aligned}$$

Then (27) follows from (14) and (15) since in (28)

$$P\{Y_k^*(z) \leq u_n, Y_k(z) > u_n\} = P\{Y_k(z) > u_n\} P\{Y_k^*(z) \leq u_n | Y_k(z) > u_n\}$$

holds.

5 Application to Indices of Complex Networks

Theorem 4 can be applied to PageRank and the Max-linear processes. These processes then have the same tail index and the same extremal index. Theorem 4 is in the agreement with statements in [5] and [14], namely, that the stationary distribution of PageRank $R = \sum_{j=1}^{N_i} A_j R_i^{(j)} + Q_i$ is regularly varying and its tail index is determined by a most heavy-tailed distributed term in the triple $(N_i, Q_i, A_i R_i^{(j)})$. This is derived if all terms in the triple are mutually independent. In contrast, Theorem 4 is valid for an arbitrary dependence structure between N_n and $\{Y_n^{(j)}\}$ as well as N_n and Q_n , and $\{N_i\}$ are not necessarily independent. The novelty of Theorem 4 is that the extremal index of both PageRank and the Max-linear processes is the same and it depends on the tail indices in the couple $(Q_i, A_i R_i^{(j)})$, irrespective of the tail index of N_i . The assumptions of both Theorem 4 and the statements in [5] and [14] do not reflect properly the complicated dependence between node ranks due to the entanglement of links in a real network. For better understanding let us consider the matrix

$$\begin{pmatrix} z_1 Y_1^{(1)} & z_2 Y_1^{(2)} & \dots & z_{N_1} Y_1^{(N_1)} & 0 & 0 & Q_1 \\ z_1 Y_2^{(1)} & z_2 Y_2^{(2)} & \dots & z_{N_1} Y_2^{(N_1)} & \dots & z_{N_2} Y_2^{(N_2)} & 0 & Q_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ z_1 Y_n^{(1)} & z_2 Y_n^{(2)} & \dots & z_{N_1} Y_n^{(N_1)} & \dots & z_{N_2} Y_n^{(N_2)} & \dots & z_{N_n} Y_n^{(N_n)} & Q_n \end{pmatrix}$$

$$((k, \theta_1) (k, \theta_2) \dots (k, \theta_{N_1}) \dots (k, \theta_{N_2}) \dots \dots (k, \theta_{N_n}) \dots (\beta, (z^*)^\beta))$$

corresponding to (16) and completed by zeros up to the maximal dimension, let's say N_n . Strings of the matrix correspond to generations of descendants of nodes with numbers $1, 2, \dots, n$. Each column may contain descendants of different nodes having the same extremal index θ_i , $i = 1, 2, \dots, N_n$. All columns apart of the last one are identically regularly varying distributed with the same tail index k . The columns are mutually independent.

In terms of some network, the conditions of Theorem 4 imply that ranks of all nodes with incoming links to a root node (i.e. its followers) are mutually independent, but followers of different nodes may be dependent and, thus, they are combined into clusters. The reciprocal of the extremal index approximates the mean cluster size, [12].

The statement (18) implies that the extremal index of PageRank is equal to $\theta(z) = (1 - c)^\beta$ if the user preference dominates (i.e. its distribution tail is heavier than the tail of ranks of followers). If the damping factor c is close to one, then $\theta(z)$ is close to zero. The latter means the huge-sized cluster of nodes around a root-node in the presence of rare teleportations. If c is close to zero, then $\theta(z)$ is close to one due to the independence of frequent teleportations. If $k < \beta$ holds, then roughly, the mean size of the cluster is determined by the consolidation of all clusters related to the followers of the underlying root.

In practice, the followers of a node may be linked and their ranks can therefore be dependent. The future work will focus on the extremal index of PageRank process when the terms $\{Y_i^{(j)}\}$ in (16) are mutually dependent.

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